# CAIE FURTHER MATHS 9231

# 2012-2024

# ROOTS OF POLYNOMIALS (FP1)

## Worked Solutions By


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* **2** - (9231-S 2012-Paper 1/3-Q8) - *Roots of polynomial equations* Good Technique To Learn !

The cubic equation $x^3 - x^2 - 3x - 10 = 0$ has roots $\alpha$, $\beta$, $\gamma$.

**(i)** Let $u = -\alpha + \beta + \gamma$. Show that $u + 2\alpha = 1$, and hence find a cubic equation having roots $-\alpha + \beta + \gamma$, $\alpha - \beta + \gamma$, $\alpha + \beta - \gamma$. [5]

**(ii)** State the value of $\alpha\beta\gamma$ and hence find a cubic equation having roots $\frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}, \frac{1}{\alpha\beta}$. [5]

$x^3 - x^2 - 3x - 10 = 0$ , Roots $\alpha, \beta, \gamma$

$\sum\alpha = 1$
$\sum\alpha\beta = -3$
$\sum\alpha\beta\gamma = 10$

(i) If $u = -\alpha + \beta + \gamma$ then $u + 2\alpha = -\alpha + \beta + \gamma + 2\alpha = \alpha + \beta + \gamma = 1$ # QED

$\Rightarrow u = 1 - 2\alpha$

If $v = \alpha - \beta + \gamma$ then $v + 2\beta = 1 \Rightarrow v = 1 - 2\beta$.
and if $w = \alpha + \beta - \gamma$ then $w + 2\gamma = 1 \Rightarrow w = 1 - 2\gamma$

$\therefore$ Each old root is related to new root via $y = 1 - 2x$

$\Rightarrow x = \frac{1-y}{2}$

$\left(\frac{1-y}{2}\right)^3 - \left(\frac{1-y}{2}\right)^2 - 3\left(\frac{1-y}{2}\right) - 10 = 0$

$(1-y)^3 - 2(1-y)^2 - 12(1-y) - 80 = 0$.

$1 - 3y + 3y^2 - y^3 - 2(1 - 2y + y^2) - 12 + 12y - 80 = 0$

$-y^3 + y^2 + 13y - 93 = 0$

$y^3 - y^2 - 13y + 93 = 0$ #

(ii) $\alpha\beta\gamma = 10$ #

New roots $u = \frac{\alpha}{\alpha\beta\gamma}$ , $v = \frac{\beta}{\alpha\beta\gamma}$ , $w = \frac{\gamma}{\alpha\beta\gamma}$

$u = \frac{\alpha}{10}$ , $v = \frac{\beta}{10}$ , $w = \frac{\gamma}{10}$.

$\therefore y = \frac{x}{10} \Rightarrow x = 10y$

$1000y^3 - 100y^2 - 30y - 10 = 0 \Rightarrow 100y^3 - 10y^2 - 3y - 1 = 0$ #

* **3** - (9231-W 2012-Paper 1/3-Q7) - *Roots of polynomial equations*

A cubic equation has roots $\alpha$, $\beta$ and $\gamma$ such that

$$\alpha + \beta + \gamma = 4,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 14,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 34.$$

Find the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [2]

Show that the cubic equation is

$$x^3 - 4x^2 + x + 6 = 0,$$

and solve this equation. [6]

$S_1 = 4 \quad S_2 = 14 \quad S_3 = 34$

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$14 = 4^2 - 2\Sigma\alpha\beta \Rightarrow \Sigma\alpha\beta = 1$ #

$x^3 - 4x^2 + x + d = 0$ ——— (1)

$S_3 - 4S_2 + S_1 + 3d = 0$

$34 - 4(14) + 4 + 3d = 0$

$\Rightarrow 3d = 18$

$\Rightarrow d = 6$

Alt:
Know $\Sigma\alpha = 4$, $\Sigma\alpha\beta = 1$
then $\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3\Sigma\alpha\beta\Sigma\alpha + 3\Sigma\alpha\beta\gamma$
$\therefore 34 = 4^3 - 3(1)(4) + 3\Sigma\alpha\beta\gamma$
$\therefore \Sigma\alpha\beta\gamma = -6.$
$\therefore$ Polynomial is
$x^3 - 4x + x + 6 = 0$

$\therefore$ Cubic eqn is

$x^3 - 4x^2 + x + 6 = 0$ # QED

$f(x) = x^3 - 4x^2 + x + 6$

$f(-1) = -1 - 4 - 1 + 6 = 0 \Rightarrow x+1$ is a factor

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ x+1 & x^3 - 4x^2 + x + 6 \\ & x^3 + x^2 \\ \hline & -5x^2 + x \\ & -5x^2 - 5x \\ \hline & 6x + 6 \end{array}$$

$f(x) = (x+1)(x-2)(x-3)$

$\therefore \alpha = -1, \beta = 2, \gamma = 3$ #